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Existence results for impulsive differential inclusions with nonlocal conditions

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ABSTRACT

In this paper, we shall establish sufficient conditions for the existence of mild solutions for nonlocal impulsive differential inclusions. On the basis of the fixed point theorems for multivalued maps and the technique of approximate solutions, new results are obtained. Examples are also provided to illustrate our results.

1. Introduction

In this paper, we shall be concerned with the following differential inclusions with nonlocal conditions and impulsive conditions:

$$\begin{cases} u'(t) \in Au(t) + F(t, u(t)), & t \in [0, b],\ t \neq t_i, \\ u(0) = g(u), \\ \Delta u(t_i) = I_i(u(t_i)), & i = 1, 2, \dots, p,\ 0 < t_1 < t_2 < \cdots < t_p < b, \end{cases} \tag{P}$$

where $A : D(A) \subseteq X \to X$ is the densely defined generator of a strongly continuous semigroup $T(\cdot)$ in a Banach space $(X, \|\cdot\|)$, F is an upper Carathéodory multifunction, $\Delta u(t_i) = u(t_i^+) - u(t_i^-)$, $u(t_i^+)$, and $u(t_i^-)$ denote the right and the left limit of u at t_i. g, and I_i are appropriate continuous functions to be specified later.

The theory of impulsive differential and partial differential equations has become an important area of investigation because of its wide use in physics, chemical technology, population dynamics, biotechnology and economics. There has been a significant development in impulsive evolution equations. We refer readers to the monographs of Benchohra et al. [1], Lakshmikantham et al. [2], and Samoilenko and Perestyuk [3], and the papers [4–11].

On the other hand, the work concerning abstract nonlocal semilinear initial-value problems was initiated by Byszewski [12,13]. They studied and obtained the existence and uniqueness of mild solutions for nonlocal differential equations without impulsive conditions in the case where Lipschitz-type conditions are satisfied. Since it is demonstrated that the nonlocal problems have better effects in applications than the classical ones, differential equations with nonlocal problems have been studied extensively in the literature. Ntouyas and Tsamatos [14,15] and Liang et al. [16] study the case where the operator semigroup $T(t)$ is compact and the single-valued functions f and g satisfy appropriate conditions. As

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operator semigroup is usually supposed to be compact. Then the main difficulty involving the nonlocal problem is how to deal with the compactness of the solution operator at zero. To fill the gap, some authors developed different methods and added conditions to F and g; see [16–23]. Among them, Aizicovici and Lee [18] discuss the existence of integral solutions for nonlocal differential inclusions when X is separable, the operator semigroup is compact and F is closed valued and lower semicontinuous in its second variable. Using the measure of noncompactness, Zhu and Li [24] get existence results for mild solutions for nonlocal problems when the evolution system is not compact.

Recently, Liang et al. [25] and Fan and Li [26] discussed the nonlocal impulsive differential equations

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)), \quad t \in [0, b],\ t \neq t_i, \\ u(0) + g(u) = u_0, \\ \Delta u(t_i) = I_i(u(t_i)), \quad i = 1, 2, \dots, p,\ 0 < t_1 < t_2 < \cdots < t_p < b, \end{cases}$$

where f and I_i satisfy the Lipschitz conditions and g is Lipschitz continuous, compact, or strongly continuous. Motivated by the above works, by means of the multivalued fixed point theorem and the method of approximate solutions, we extend the results shown in [25,26] to the differential inclusions scenario and generalize the conditions on F and g, that is, the compactness conditions and Lipschitz conditions are not necessary in our results.

This paper is organized as follows. In Section 2, we recall some concepts and facts concerning multivalued analysis. In Section 3, using the method of approximate solutions, we give the existence results for problem (P) when the nonlocal function g has no compactness conditions or Lipschitz conditions. Finally, examples are presented to illustrate the application of our results.

2. Preliminaries

Let X and Y be two Hausdorff topological spaces. We use the following notation: $P(Y) = \{A \in 2^Y : A \neq \varnothing\}$, $P_{cl}(Y) = \{A \in P(Y) : A \text{closed}\}$, $P_b(Y) = \{A \in P(Y) : A \text{bounded}\}$, $P_c(Y) = \{A \in P(Y) : A \text{convex}\}$, $P_{cp}(Y) = \{A \in P(Y) : A \text{compact}\}$, $P_{cp,c}(Y) = \{A \in P(Y) : A \in P_{cp}(Y) \bigcap P_c(Y)\}$. A multivalued map $F : X \to P(Y)$ is said to be convex (closed) valued if $F(x)$ is convex (closed) in Y for all $x \in X$. F is said to be compact if $F(B)$ is relatively compact for every $B \in P_b(X)$.

$F : X \to P(Y)$ is said to be upper semicontinuous (u.s.c.) on X if for each $x_0 \in X$ the set $F(x_0)$ is a nonempty, closed subset of Y, and if for each open subset K of Y containing $F(x_0)$, there exists an open neighborhood Γ of x_0 such that $F(\Gamma) \subseteq K$.

The following conclusions are useful for getting the upper semicontinuity of a multifunction F. Assume that $D \subset X$ and Fx is closed for all $x \in D$; then:

(i) If F is u.s.c. and D is closed, then F has a closed graph (i.e., $x_n \to x$ and $y_n \to y$, $y_n \in F(x_n)$, imply $y \in F(x)$).
(ii) If $\overline{F(D)}$ is compact and D is closed, then F is u.s.c. if and only if F has a closed graph.

Throughout this paper, let $(X, \|\cdot\|)$ be a real Banach space. We denote by $C([0, b]; X)$ the space of X-valued continuous functions on $[0, b]$ with the norm $\|x\| = \sup\{\|x(t)\|, t \in [0, b]\}$ and by $L^1([0, b]; X)$ the space of X-valued Bochner integrable functions on $[0, b]$ with the norm $\|f\|_{L^1} = \int_0^b \|f(t)\| \, dt$. In order to define the mild solution of impulsive problem (P), we introduce the set $PC([0, b]; X) = \{u : [0, b] \to X : u \text{ is continuous at } t \neq t_i \text{ and left continuous at } t = t_i, \text{ and the right limit } u(t_i^+) \text{ exists}, i = 1, \dots, p\}$. It is easy to verify that $PC([0, b]; X)$ is a Banach space with the norm $\|u\|_{PC} = \sup\{\|u(t)\|, t \in [0, b]\}$.

We set $S_F(u) := \{f \in L^1([0, b]; X) : f(t) \in F(t, u(t)) : \text{a.e. on } [0, b]\}$, and say that F has a fixed point if there is $x \in X$ such that $x \in F(x)$.

Definition 2.1. A function $u \in PC([0, b]; X)$ is said to be a mild solution of problem (P) if

$$u(t) = T(t)g(u) + \int_0^t T(t - s)f(s) \, ds + \sum_{0<t_i<t} T(t - t_i)I_i(u(t_i)),$$

for all $t \in [0, b]$, where $f \in S_F(u)$.

We first give the assumptions for the function F.

(HF) F is an upper Carathéodory multifunction, i.e., for every $x \in X$ the multifunction $F(\cdot, x) : [0, b] \to P_{cp,c}(X)$ admits a strongly measurable selector; for a.e. $t \in [0, b]$ the multifunction $F(t, \cdot) : X \to P_{cp,c}(X)$ is u.s.c.. And for each fixed $u \in PC([0, b]; X)$, the set $S_F(u)$ is nonempty.

Our main results are based on the following lemmas.

Lemma 2.2 ([27]). *Let X be a Banach space and F a multifunction satisfying assumption* (HF). *Let $\Gamma : L^1([0, b]; X) \to C([0, b]; X)$ be a linear continuous mapping. Then the operator*

$$\Gamma \circ S_F : C([0, b]; X) \to P_{cl,c}(C([0, b]; X)), \qquad x \to (\Gamma \circ S_F)(x) := \Gamma(S_F(x))$$

is a closed graph operator in $C([0, b]; X) \times C([0, b], X)$.

Lemma 2.3 ([28]). *Let D be a nonempty, closed, convex subset of a completely Hausdorff locally convex linear topological space and let $Q : D \to P(D)$ be an upper semicontinuous, compact map with $Q(x)$ a nonempty, closed, convex subset of D. Then Q has a fixed point in D.*

3. The main results

In this section, by using the method of approximate solutions, we give the existence results for problem (P). Indeed, we only suppose that the nonlocal function g is continuous and deal with the difficulty of the compactness of the solution operator at zero neatly.

Let r be a positive constant, and define $B_r = \{x \in X : \|x\| \le r\}$, $W_r = \{u \in PC([0, b]; X) : u(t) \in B_r, \forall t \in [0, b]\}$. Here, we list the following hypotheses:

(HA) $A : D(A) \subseteq X \to X$ generates a compact strongly continuous operator semigroup $\{T(t) : t \ge 0\}$, that is, $T(t)$ is compact for $t > 0$. Moreover, there exists a positive constant $M > 0$ such that $M = \sup_{0 \le t \le b} \|T(t)\|$ (see [29]).

(Hg) $g : PC([0, b]; X) \to X$ is a continuous mapping which maps W_r into a bounded set. And there is a $\delta = \delta(r) \in (0, t_1)$ such that $g(u) = g(v)$ for any $u, v \in W_r$, with $u(s) = v(s)$, $s \in [\delta, b]$.

(HI) $I_i : X \to X$ is a compact operator for $i = 1, 2, \ldots, p$.

Theorem 3.1. *Assume that the hypotheses* (HF), (HA), (Hg), (HI) *are satisfied; then the nonlocal impulsive problem* (P) *has at least one mild solution on* $[0, b]$, *provided that there exists a constant* $r > 0$ *such that*

$$M \left[\sup_{u \in W_r} \|g(u)\| + \sup_{u \in W_r} \{\|f\|_{L^1} : f \in S_F(u)\} + \sup_{u \in W_r} \sum_{i=1}^{p} \|I_i(u(t_i))\| \right] \le r. \tag{1}$$

In order to prove the above theorem, we need some lemmas.

Lemma 3.2. *Suppose that conditions* (HF) *and* (HA) *are satisfied. Suppose that* $W_r = \{x \in PC([0, b]; X) : \|x\| \le r\}$. *Then the mapping* $G : W_r \to P(C([0, b]; X))$ *defined by*

$$(Gu)(t) = \left\{ v \in C([0, b]; X) : v(t) = \int_0^t T(t - s)f(s)\,\mathrm{d}s,\ f \in S_F(u) \right\},$$

is compact.

Proof. It is enough to show that GW_r is relatively compact in $C([0, b]; X)$. Firstly, we show that, for each $t \in [0, b]$, $u \in W_r$, the set $(Gu)(t)$ is relatively compact in X. If $t = 0$, then $(Gu)(0) = 0$. Suppose that $t \in (0, b]$ and $\varepsilon \in (0, t)$, and

$$\begin{aligned}(G^{\varepsilon}u)(t) &:= \left\{ \int_0^{t-\varepsilon} T(t - s)f(s)\,\mathrm{d}s : f \in S_F(u) \right\} \\ &= \left\{ T(\varepsilon) \int_0^{t-\varepsilon} T(t - \varepsilon - s)f(s)\,\mathrm{d}s : f \in S_F(u) \right\}\end{aligned}$$

is relatively compact in X since $T(\varepsilon)$ is compact. Then, as

$$(G^{\varepsilon}u)(t) \to (Gu)(t), \quad \text{as } \varepsilon \to 0,$$

we conclude that, for each $t \in [0, b]$, $(Gu)(t)$ is relatively compact in X by using the total boundedness.

Next, we prove the equicontinuity of GW_r. We suppose that $0 \le t_1 < t_2 \le b$, $f \in S_F(u)$, $u \in W_r$ and obtain

$$\begin{aligned}&\left\| \int_0^{t_2} T(t_2 - s)f(s)\,\mathrm{d}s - \int_0^{t_1} T(t_1 - s)f(s)\,\mathrm{d}s \right\| \\ &= \left\| \int_0^{t_1} [T(t_2 - s) - T(t_1 - s)]f(s)\,\mathrm{d}s + \int_{t_1}^{t_2} T(t_2 - s)f(s)\,\mathrm{d}s \right\| \\ &\le \int_0^{t_1} \|T(t_2 - s) - T(t_1 - s)\|\,\|f(s)\|\,\mathrm{d}s + M \int_{t_1}^{t_2} \|f(s)\|\,\mathrm{d}s.\end{aligned} \tag{2}$$

If $t_1 = 0$, then the right hand side of (2) can be made small when t_2 is small independently of $u \in W_r$. If $t_1 > 0$, then we can find a small number $\varepsilon > 0$ with $t_1 - \varepsilon > 0$; then it follows from (2) that

$$\begin{aligned}&\int_0^{t_1} \|T(t_2 - s) - T(t_1 - s)\|\,\|f(s)\|\,\mathrm{d}s + M \int_{t_1}^{t_2} \|f(s)\|\,\mathrm{d}s \\ &\le \int_0^{t_1-\varepsilon} \|T(t_2 - s) - T(t_1 - s)\|\,\|f(s)\|\,\mathrm{d}s + 2M \int_{t_1-\varepsilon}^{t_1} \|f(s)\|\,\mathrm{d}s + M \int_{t_1}^{t_2} \|f(s)\|\,\mathrm{d}s.\end{aligned} \tag{3}$$

Here, $T(t)$ is compact for $t > 0$. Thus $T(t)$ is operator norm continuous for $t > 0$. Therefore, we have

$$\int_0^{t_1-\varepsilon} \|T(t_2 - s) - T(t_1 - s)\|\,\|f(s)\|\,\mathrm{d}s \to 0, \quad \text{as } t_1 \to t_2,$$

uniformly for all $f \in S_F(u)$ and $u \in W_r$. Then, from (3), we see that $\{(Gu)(\cdot) : u \in W_r\}$ is equicontinuous. By the well-known Ascoli–Arzela theorem, we know that GW_r is relatively compact in $C([0, b]; X)$. This completes the proof. □

For fixed $n \geq 1$, we consider the following approximate problem:

$$\begin{cases} u'(t) \in Au(t) + F(t, u(t)), & t \in [0, b],\ t \neq t_i, \\ u(0) = T\left(\frac{1}{n}\right) g(u), \\ \Delta u(t_i) = I_i(u(t_i)), & i = 1, 2, \ldots, p,\ 0 < t_1 < t_2 < \cdots < t_p < b. \end{cases} \tag{P$_n$}$$

Lemma 3.3. *Assume that all of the conditions in Theorem 3.1 are satisfied. Then for any $n \geq 1$, the nonlocal problem (P$_n$) has at least one mild solution on $[0, b]$.*

Proof. For fixed $n \geq 1$, set $Q_n : PC([0, b]; X) \to P(PC([0, b]; X))$ defined by

$$Q_n(u) = \left\{ v \in PC([0, b]; X) : v(t) = T(t)T\left(\frac{1}{n}\right) g(u) + \int_0^t T(t-s)f(s)\,ds + \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i)),\ f \in S_F(u) \right\},$$

with

$$(Q_{n1}u)(t) = T(t)T\left(\frac{1}{n}\right) g(u),$$

$$(Q_{n2}u)(t) = \left\{ v \in C([0, b]; X) : v(t) = \int_0^t T(t-s)f(s)\,ds,\ f \in S_F(u) \right\},$$

$$(Q_{n3}u)(t) = \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i)).$$

It is easy to see that the fixed point of Q_n is the mild solution of nonlocal problem (P$_n$).

From (1), we know that the mapping Q_n maps W_r into itself. Firstly, we prove that Q_n has a closed graph on $PC([0, b]; X)$ with closed convex values. In fact, it is easy to check that Q_n has convex values. Now we show that Q_n has a closed graph. Suppose that $(u_m)_{m\in\mathbb{N}}, (v_m)_{m\in\mathbb{N}} \subset PC([0, b]; X)$, $u_m \to u$, $v_m \in Q_n(u_m)$, $v_m \to v$, in $PC([0, b]; X)$. Then there exists a sequence $\{f_m\}_{m=1}^{\infty} \subset L^1([0, b]; X)$, $f_m \in S_F(u_m)$ for $m \geq 1$, such that

$$v_m(t) = T(t)T\left(\frac{1}{n}\right) g(u_m) + \int_0^t T(t-s)f_m(s)\,ds + \sum_{0<t_i<t} T(t-t_i)I_i(u_m(t_i)), \tag{4}$$

for all $t \in [0, b]$. Consider the linear operator $\Gamma : L^1([0, b]; X) \to C([0, b]; X)$ defined as $(\Gamma f)(t) = \int_0^t T(t-s)f(s)\,ds$. Obviously, Γ is linear and continuous. From Eq. (4), we have

$$v_m(\cdot) - T(\cdot)T\left(\frac{1}{n}\right) g(u_m) - \sum_{0<t_i<\cdot} T(\cdot - t_i)I_i(u_m(t_i)) \in \Gamma \circ S_F(u_m).$$

Then from Lemma 2.2, we get that $\Gamma \circ S_F(\cdot)$ is a closed graph operator. Since $u_m \to u$ and $v_m \to v$, we obtain that

$$v(\cdot) - T(\cdot)T\left(\frac{1}{n}\right) g(u) - \sum_{0<t_i<\cdot} T(\cdot - t_i)I_i(u(t_i)) \in \Gamma \circ S_F(u),$$

from the continuity of $T(t), g, I_i$. That is,

$$v(t) - T(t)T\left(\frac{1}{n}\right) g(u) - \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i)) = \int_0^t T(t-s)f(s)\,ds,$$

for some $f \in S_F(u)$. Therefore, Q_n has a closed graph. Moreover, this implies that Q_n has closed values on $PC([0, b]; X)$.

Next, we show that Q_n is a compact operator. Since $T\left(\frac{1}{n}\right)$ is compact at $n \geq 1$ and the assumption (Hg) holds, it is obvious that the set $\{T\left(\frac{1}{n}\right)T(t)g(u) : u \in W_r\}$ is relatively compact in X. Then, for $0 \leq s < t \leq b$, we have

$$\left\| T(t)T\left(\frac{1}{n}\right) g(u) - T(s)T\left(\frac{1}{n}\right) g(u) \right\| = \left\| [T(t) - T(s)]T\left(\frac{1}{n}\right) g(u) \right\|.$$

Thus, the functions in $Q_{n1}W_r$ are equicontinuous due to the compactness of $T\left(\frac{1}{n}\right)$ and the strong continuity of $T(\cdot)$. Now an application of the Ascoli–Arzela theorem justifies the assertion that Q_{n1} is a compact operator. Note that Q_{n2} is compact due to Lemma 3.2. To prove the compactness of Q_{n3}, note that

$$(Q_{n3}u)(t) = \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i)) = \begin{cases} 0, \quad t \in [0, t_1], \\ T(t-t_1)I_1(u(t_1)), \quad t \in (t_1, t_2], \\ \cdots \\ \sum_{i=1}^{p} T(t-t_i)I_i(u(t_i)), \ t \in (t_p, b], \end{cases}$$

and that interval $[0, b]$ is divided into finite subintervals by t_i, $i = 1, 2, \ldots, p$, so we only need to prove that

$$W = \{T(\cdot - t_1)I_1(u(t_1)) : \cdot \in [t_1, t_2], u \in W_r\}$$

is relatively compact in $C([t_1, t_2]; X)$, as the cases for other subintervals are the same.

As I_i is compact and $T(t)$ is continuous, the set $\{T(t-t_1)I_1(u(t_1)) : u \in W_r\}$ is relatively compact in X for each $t \in [t_1, t_2]$. Next, for $t_1 \le s < t \le t_2$, we have, using the semigroup property,

$$\begin{aligned} \|T(t-t_1)I_1(u(t_1)) - T(s-t_1)I_1(u(t_1))\| &= \|T(s-t_1)[T(t-s) - T(0)]I_1(u(t_1))\| \\ &\le M\|[T(t-s) - T(0)]I_1(u(t_1))\|. \end{aligned}$$

Thus, W is equicontinuous due to the compactness of I_i and the strong continuity of $T(\cdot)$. By the Ascoli–Arzela theorem, we conclude that W is relatively compact in $C([t_1, t_2]; X)$. Therefore, Q_{n3} is a compact operator. So we get that $\{Q_n u : u \in W_r\}$ is relatively compact in $PC([0, b]; X)$. As Q_n has a closed graph, from the property of multivalued maps, we have that Q_n is u.s.c.

Finally, due to Lemma 2.3, Q_n has at least one fixed point $u_n \in W_r$, and u_n is a mild solution of nonlocal impulsive problem ($\mathrm{P_n}$). This completes the proof. □

Now, we define the solution set D by

$$D = \{u_n \in PC([0, b]; X) : u_n \in Q_n u_n, \ n \ge 1\}.$$

Proof of Theorem 3.1. Consider the solution set D; we will prove that it is relatively compact in $PC([0, b]; X)$. To this end, suppose that

$$u_n(t) = T(t)T\left(\frac{1}{n}\right)g(u_n) + \int_0^t T(t-s)f_n(s)\,\mathrm{d}s + \sum_{0<t_i<t} T(t-t_i)I_i(u_n(t_i)), \tag{5}$$

where $f_n \in S_F(u_n)$, with

$$\begin{aligned} (Q_{n1}u_n)(t) &= T(t)T\left(\frac{1}{n}\right)g(u_n), \\ (Q_{n2}u_n)(t) &= \int_0^t T(t-s)f_n(s)\,\mathrm{d}s, \\ (Q_{n3}u_n)(t) &= \sum_{0<t_i<t} T(t-t_i)I_i(u_n(t_i)), \end{aligned}$$

for all $t \in [0, b]$.

Firstly, we prove that $D(t) = \{u_n(t) : n \ge 1\}$ is relatively compact in X for each $t \in (0, b]$. By the compactness of $T(t)$, I_i and the boundedness of $\left\{T\left(\frac{1}{n}\right)g(u_n)\right\}$, we get that $\{(Q_{n1}u_n)(t) + (Q_{n3}u_n)(t) : n \ge 1\}$ is relatively compact for each $0 < t < b$. Moreover, from Lemma 3.2, we get that $\{(Q_{n2}u_n)(t) : n \ge 1\}$ is relatively compact in X. And so is the set $D(t)$ for each $t \in (0, b]$.

Next, we show that D is equicontinuous on $(0, t_1]$ and each $\overline{J_i} = [t_i, t_{i+1}]$, $i = 1, 2, \ldots, p$. If $t \in (0, t_1]$, then for any $\varepsilon, h > 0$ such that $t - \varepsilon > 0$ and $t + h \le t_1$, we have

$$\begin{aligned} \|u_n(t+h) - u_n(t)\| &= \left\|[T(t+h) - T(t)]T\left(\frac{1}{n}\right)g(u_n)\right\| + \int_0^t \|[T(t+h-s) - T(t-s)]f_n(s)\|\,\mathrm{d}s \\ &\quad + \int_t^{t+h} \|T(t+h-s)f_n(s)\|\,\mathrm{d}s, \\ &\le \left\|[T(t+h) - T(t)]T\left(\frac{1}{n}\right)g(u_n)\right\| + \int_0^{t-\varepsilon} \|[T(t+h-s) - T(t-s)]f_n(s)\|\,\mathrm{d}s \\ &\quad + 2M\int_{t-\varepsilon}^t \|f_n(s)\|\,\mathrm{d}s + M\int_t^{t+h} \|f_n(s)\|\,\mathrm{d}s. \end{aligned}$$

As $T(t)$ is operator norm continuous at $t > 0$ and hypothesis (Hg) holds, we have

$$\left\|[T(t+h)-T(t)]T\left(\frac{1}{n}\right)g(u_n)\right\| \to 0, \quad \text{as } h \to 0,$$

uniformly for all $u_n \in D$ and

$$\int_0^{t-\varepsilon} \|[T(t+h-s)-T(t-s)]f_n(s)\| \, ds \to 0, \quad \text{as } h \to 0,$$

uniformly for all $f_n \in S_F(u_n)$ and $u_n \in D$. Therefore we get that D is equicontinuous on $(0, t_1]$.

If $t \in \overline{J_1} = [t_1, t_2]$, then for a positive constant $h > 0$ such that $t + h \le t_2$, we have

$$\begin{aligned}\|u_n(t+h)-u_n(t)\| = {} & \left\|[T(t+h)-T(t)]T\left(\frac{1}{n}\right)g(u_n)\right\| + \int_0^t \|[T(t+h-s)-T(t-s)]f_n(s)\| \, ds \\ & + \int_t^{t+h} \|T(t+h-s)f_n(s)\| \, ds + \|[T(t+h-t_1)-T(t-t_1)]I_1(u_n(t_1))\|.\end{aligned}$$

From the above discussion for $t \in (0, t_1]$, we only need to prove that, for $t \in [t_1, t_2]$,

$$\|[T(t+h-t_1)-T(t-t_1)]I_1(u_n(t_1))\| \to 0, \quad \text{as } h \to 0,$$

independently of $\underline{u_n} \in D$, which can be obtained by the compactness of I_i and the strong continuity of $T(t)$. So D is equicontinuous on $\overline{J_1}$. And the same idea can be used for other subintervals $\overline{J_i}$, $i = 2, \ldots, p$.

Suppose that $\delta \in (0, t_1)$ and

$$u_n^*(t) = \begin{cases} u_n(\delta), & 0 \le t \le \delta, \\ u_n(t), & \delta \le t \le b. \end{cases}$$

It is easy to check that $\{u_n^* : n \ge 1\}$ is relatively compact in $PC([0, b]; X)$. Without loss of generality, there exists $u^* \in W_r$ with $\lim_{n\to\infty} u_n^* = u^*$. By (Hg), we get that $g(u_n) = g(u_n^*) \to g(u^*)$ as $n \to \infty$. Then

$$T(t)T\left(\frac{1}{n}\right)g(u_n) = T(t)T\left(\frac{1}{n}\right)g(u_n^*) \to T(t)g(u^*), \quad \text{as } n \to \infty. \tag{6}$$

From Lemma 3.2, we have that $Q_{n2}u_n$ is relatively compact. Without loss of generality (otherwise, we can pass to the subsequence), we assume that

$$\int_0^{\cdot} T(\cdot - s)f_n(s) \, ds \to \omega \in W_r, \quad \text{as } n \to \infty, \text{ for } \cdot \in [0, b], \tag{7}$$

where $f_n \in S_F(u_n)$. Thus, by (6), (7) and the compactness of I_i, invoking (5), we know that $\{u_n : n \ge 1\}$ is a convergent sequence in a closed ball W_r. So we can suppose that

$$\lim_{n\to\infty} u_n = u, \quad u \in W_r. \tag{8}$$

It follows from (5) that

$$u_n(\cdot) - T(\cdot)T\left(\frac{1}{n}\right)g(u_n) - \sum_{0<t_i<\cdot} T(\cdot - t_i)I_i(u_n(t_i)) \in \varGamma \circ S_F(u_n),$$

where $\varGamma : L^1([0, b]; X) \to C([0, b]; X)$ is defined as $(\varGamma f)(t) = \int_0^t T(t-s)f(s) \, ds$. $\varGamma \circ S_F$ is a closed graph operator due to Lemma 2.2. Then supposing that $n \to \infty$, we have

$$u(\cdot) - T(\cdot)g(u^*) - \sum_{0<t_i<\cdot} T(\cdot - t_i)I_i(u(t_i)) \in \varGamma \circ S_F(u),$$

combining (6)–(8).

That is, there exists some $f \in S_F(u)$ such that

$$u(t) = T(t)g(u^*) + \int_0^t T(t-s)f(s) \, ds + \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i)), \quad t \in [0, b]. \tag{9}$$

Moreover, $g(u) = g(u^*)$, since

$$u(t) = \lim_{n\to\infty} u_n(t) = \lim_{n\to\infty} u_n^*(t) = u^*(t), \quad t \in [\delta, b],$$

by the definition of g. This leads to the conclusion, from (9), that u is a mild solution of nonlocal impulsive problem (P). This completes the proof. □

Corollary 3.4. *Assume that hypotheses* (HF), (HA), (Hg), (HI) *hold true for each* $r > 0$. *Suppose that* $|F(t,x)| := \sup\{\|y\| : y \in F(t,x)\}$. *If*

$$\frac{|F(t,x)|}{\|x\|} \to 0, \quad \|x\| \to \infty, \tag{10}$$

$$\frac{\|g(u)\|}{\|u\|_{PC}} \to 0, \quad \|u\|_{PC} \to \infty, \tag{11}$$

$$\frac{\sum_{i=1}^{p} \|I_i(x)\|}{\|x\|} \to 0, \quad \|x\| \to \infty, \tag{12}$$

then the nonlocal impulsive problem (P) *has at least one mild solution in* $PC([0,b];X)$.

Remark 3.5. The technique of approximate solutions plays a key role in the proof of Theorem 3.1, which enables us to deal with the difficulty involving the nonlocal function g, i.e., the compactness problem of the solution operator at zero. This method is also used in [26,30], where the existence of mild solutions is obtained when the semigroup is compact. As we do not need special conditions on F and g, our results, even in the single-valued case, generalize and extend some corresponding results in this field and have wider applications.

4. Examples

In this section, we give two examples to illustrate our results.

Example 4.1. In many studies of the nonlocal Cauchy problem, such as [12,31,32], the mapping g is given by $g(s_1, \dots, s_q, u(s_1), \dots, u(s_q))$, where $0 < s_1 < s_2 < \cdots < s_q \le b$. For example, in [31],

$$g(s_1, \dots, s_q, u(s_1), \dots, u(s_q)) = \sum_{j=1}^{q} c_j u(s_j), \tag{13}$$

for some given constant c_j. This allows measurements to be made at $t = 0, s_1, \dots, s_q$ rather than just at $t = 0$ and more information can be obtained. Note that g in (13) satisfies (Hg) with $\delta = s_1$.

We consider the following nonlocal impulsive problem:

$$\begin{cases} u'(t) \in Au(t) + F(t,u(t)), \quad t \in [0,b],\ t \ne t_i, \\ u(0) = g(s_1, \dots, s_q, u(s_1), \dots, u(s_q)) = \sum_{j=1}^{q} c_j u(s_j), \\ \Delta u(t_i) = I_i(u(t_i)), \quad i = 1, 2, \dots, p,\ 0 < t_1 < t_2 < \cdots < t_p < b. \end{cases} \tag{E1}$$

Assume that hypotheses (HF), (HA) and (HI) are satisfied. Then, according to Theorem 3.1, problem (E1) has at least one mild solution in $PC([0,b];X)$ provided that

$$M\left[\sum_{j=1}^{q} |c_j| r + \sup_{u \in W_r} \{\|f\|_{L^1} : f \in S_F(u)\} + \sup_{u \in W_r} \sum_{i=1}^{p} \|I_i(u(t_i))\|\right] \le r$$

holds true for some $r > 0$.

Example 4.2. Consider the following partial differential system with nonlocal conditions and impulsive conditions:

$$\begin{cases} \frac{\partial}{\partial t}\omega(t,x) \in \frac{\partial^2}{\partial x^2}\omega(t,x) + F(t,\omega(t,x)), \quad 0 \le t \le b,\ 0 \le x \le \pi, \\ \omega(t,0) = \omega(t,\pi) = 0, \\ \omega(t_i^+, x) - \omega(t_i^-, x) = I_i(\omega(t_i, x)), \quad i = 1, 2, \dots, p, \\ \omega(0,x) + \sum_{j=1}^{q} c_j \sqrt[3]{\omega(s_j, x)} = u_0(x), \end{cases} \tag{E2}$$

where $u_0 \in X = L^2[0,\pi]$, c_j are given real numbers for $j = 1, 2, \dots, q$.

We consider the operator $A : D(A) \subseteq X \to X$ defined by $Az = z''$, with

$$D(A) = \{z \in X : z, z' \text{ are absolutely continuous},\ z'' \in X, z(0) = z(\pi) = 0\}.$$

From [29], we know that A generates a compact C_0-semigroup $T(t)$. This implies that A satisfies the condition (HA).

Now, we assume that:

(1) $F : [0, b] \times X \to P(X)$ is a multifunction defined by

$$F(t, z)(x) = F(t, z(x)), \quad 0 \le t \le b,\ 0 \le x \le \pi,$$

and (HF) and (10) hold true.

(2) $I_i : X \to X$ is a compact operator and satisfies (12), for $i = 1, 2, \ldots, p$.

(3) $g : PC([0, b]; X) \to X$ is a continuous function defined by

$$g(u)(x) = u_0(x) - \sum_{j=1}^{q} c_j \sqrt[3]{u(s_j)(x)}, \quad 0 \le x \le \pi,$$

where $u(s)(x) = \omega(s, x), 0 \le x \le \pi$.

Under the above assumptions, the partial differential system (E2) can be reformulated as the abstract problem (P). All the conditions in Corollary 3.4 are satisfied. Then the problem (E2) has at least one mild solution in $PC([0, b]; X)$.

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